ON THE DIRECT SUMS OF MINC_{11} AND MAXC_{11} MODULES

HAGIM RAYALONG

A THESIS APPROVED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE IN
MATEMATTICS AND MATHEMATICS EDUCATION
THAKSIN UNIVERSITY
2018

THESIS CERTIFICATION

THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS AND MATHEMATICS EDUCATION

THAKSIN UNIVERSITY

Thesis Title : On the Direct Sums of MaxC_{11} and MinC_{11} Modules

Student's Name : Mr. Hagim Rayalong

THESIS ADVISOR

(Asst. Prof. Dr. Sarapee Chairat)
Major - Advisor

(Dr. Nguyen Dang Hoa Nghiem)
Co - Advisor

THESIS DEFENCE COMMITTEE

(Dr. Nguyen Van Sanh)
Chair

(Asst. Prof. Dr. Sarapee Chairat)
Member

(Dr. Nguyen Dang Hoa Nghiem)
Member

Thaksin University Approved This Thesis in Partial Fulfillment of The Requirements for The Degree of Master of Science in Mathematics and Mathematics Education

(Dr. Wallapa Choeibuakaew)
Dean of Graduate School
February 2019

บทคัดย่อ

ชื่อวิทยานิพนธ์ : ผลบวกตรงของมอดูล C_{11} เล็กสุดและมอดูล C_{11} ใหญ่สุด

ชื่อ-ชื่อสกุลผู้ทำวิทยานิพนธ์ : นายฮากีม ระยะหลง

อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ผู้ช่วยศาสตราจารย์ ดร.สารภี ไชยรัตน์ และ Dr. Nguyen Dang Hoa Nghiem

ปริญญาและสาขาวิชา : ปริญญาวิทยาศาสตรมหาบัณฑิต สาขาวิชาคณิตศาสตร์และคณิตศาสตรศึกษา

ปีการศึกษาที่สำเร็จ : 2561

ในการวิจัยนี้ได้ให้นิยามเกี่ยวกับการมีสมบัติมอดูล C_{11} เล็กสุดและมอดูล C_{11} ใหญ่สุดของมอดูล M บนริง R ที่มีสมบัติเปลี่ยนหมู่พร้อมเอกลักษณ์ มอดูล M กล่าวว่ามีสมบัติมอดูล C_{11} เล็กสุด ถ้าทุกๆมอดูลย่อยเล็กสุดของ M สามารถหาส่วนเติมเต็มที่มีสมบัติส่วนของผลบวกตรงใน M และมอดูล M มีสมบัติมอดูล C_{11} ใหญ่สุด ถ้าทุกๆมอดูลย่อยใหญ่สุดของ M สามารถหาส่วนเติมเต็มที่มีสมบัติส่วนของผลบวกตรงใน M ผลลัพธ์ที่ได้คือถ้าทุกๆผลบวกตรงของมอดูล M มีสมบัติ C_{11} เล็กสุดแล้วมอดูล M มีสมบัติ C_{11} เล็กสุด และสมบัติมอดูล C_{11} ใหญ่สุดก็สามารถพิสูจน์ได้เช่นกัน นอกจากนี้เรายังได้พิสูจน์ว่าถ้าทุกๆส่วนของผลบวกตรงของมอดูล M มีสมบัติ C_{11} เล็กสุดแล้วมอดูล M มีสมบัติ C_{11} เล็กสุด และสมบัติมอดูล C_{11} ใหญ่สุดก็สามารถพิสูจน์ได้เช่นกัน ให้ $M = M_1 \oplus M_2$ เป็นมอดูลที่มีสมบัติมอดูล C_{11} เล็กสุดโดยที่ส่วนของผลบวกตรง K ใดๆของมอดูล M มีสมบัติ $K \cap M_2 = 0$ และ $K \oplus M_2$ มีสมบัติเป็นส่วนของผลบวกตรงใน M แล้ว M_1 มีสมบัติ C_{11} เล็กสุด

ถ้า M มีสมบัติก่อกำเนิดจำกัด คล้ายคลึงเชิงภาพฉาย และก่อกำเนิดในตัวเองแล้ว M มีสมบัติมอดูล C_{11} ใหญ่สุด (C_{11} เล็กสุด) ก็ต่อเมื่อเอนโดมอร์ฟิซึมริง S มีสมบัติริง C_{11} ใหญ่สุด (C_{11} เล็กสุด)

ABSTRACT

Thesis Title : On the Direct Sums of MinC_{11} and MaxC_{11} Modules
Student's Name : Mr. Hagim Rayalong
Advisory Committee : Asst. Prof. Dr. Sarapee Chairat and
Dr. Nguyen Dang Hoa Nghiem
Degree and Program : Master of Science in Mathematics and Mathematics Education
Academic Year : 2018

In this thesis, we defined minC_{11} and maxC_{11} modules M over an associative ring R with identity. An R-module M is said to be a minC_{11} module, if every minimal submodule has a complement which is a direct summand of M. An R-module M is said to be a maxC_{11} module, if every maximal submodule has a complement which is a direct summand of M. Then any direct sum of modules with minC_{11} satisfies minC_{11} and any direct sum of modules with maxC_{11} satisfies maxC_{11}. Furthermore, we prove that any direct sum of modules with minC_{11} (maxC_{11}) satisfies minC_{11} (maxC_{11}). Let $M = M_1 \oplus M_2$ be a minC_{11}-module such that for every direct summand K of M $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand if M. Then M_1 is a minC_{11}-module.

Moreover, if M is a finitely generated, quasi-projective right R-module which is a self-generator, then M is a maxC_{11} (minC_{11}) module if and only if the endomorphism ring S of a right R-module M is a right maxC_{11} (minC_{11}) ring.

ACKNOWLEDGMENTS

First of all I would like to thank my advisor Asst. Prof. Dr. Sarapee Chairat and Dr. Nguyen D. Hoa Nghiem, my co-advisor for guidance, encouragement and support throughout the process of this work.

I would like to express my gratitude to Department of Mathematics and Mathematics Education, Thaksin University, for providing me the necessary facilities.

I would like to thank the Graduate School of Thaksin University for the financial support.

Finally, I own unending gratitude to my family and all my teachers who filled in the knowledge and whose support and encourage me to finish my education.

Hagim Rayalong

CONTENTS

CHAPTER 1

INTRODUCTION

Mohamed and Muller (1990 : 12-37) introduced the concept of extending module, where *R*-module *M* is called an extending (or *CS*), if every submodule is essential in a direct summand of *M*. Equivalently, *M* is an extending, if and only if every closed submodule is a direct summand. Later, Dung, Huynh, Smith, and Wisbauer, studied extending modules and found many properties of extending modules. From that time, characterizations and properties of certain extending modules have become interesting and important to researchers in this area.

There are many papers concerned with the generalization of *CS*-module, an important tool in this is the notion of min*CS* and max*CS* modules. Hazmi introduced min*CS* and max*CS* modules following that an *R*-module *M* is called min*CS* (max*CS*) if every minimal submodule (every maximal submodule with nonzero right annihilator) is a direct summand of *M*. Hadi and Majeed (2012a : 1-13) studied min*CS* (max) *CS* modules. They proved that, if *R* is a nonsingular ring then *R* is a max*CS* ring if and only if *R* is a min*CS* ring. Later they proved that a direct summand of a min*CS* (max*CS*) module is a min*CS* (max*CS*) module, but the converse is not true in general.

As we know the direct sum of two *CS*-modules is not a *CS*-module. One of the most interesting questions concerning *CS*-modules is when a (finite or infinite) direct sum of *CS*-modules is also a *CS*-module. Smith and Tercan, (1993 : 1809-1847) introduced C_{11}-modules defined as follows : a module *M* satisfies C_{11} if every submodule has a complement which is a direct summand of *M*, i.e., for each submodule *N* of *M* there exists a direct summand *K* of *M* such that *K* is a complement of *N* in *M*. A C_{11}-module was defined as a general of *CS*-modules. Then we would like to work on minC_{11} and maxC_{11} modules.

For this study, the reseacher shall defined the definition of minC_{11} and maxC_{11} modules and studied some properties of minC_{11} and maxC_{11} modules. Moreover, we shall find relations of minC_{11} and maxC_{11} modules and its endomorphism rings.

Objective of the Study

1) to define the definition of minC_{11} and maxC_{11} modules ;

2) to prove that any direct sum of module with minC_{11} (maxC_{11}) satisfies minC_{11} (maxC_{11}) ;

3) to give the conditions to the direct summand of minC_{11} module satisfies minC_{11}, and

4) to find relations of minC_{11} (maxC_{11}) modules and their endomorphism rings.

Scope and Limitation

Throughout this study, all rings are associative with identity and all modules are unitary right R-modules. In this study, we shall define the definition of minC_{11} and maxC_{11} modules. The focus of our discussion in this note is mainly on the direct sum of minC_{11} and maxC_{11} modules. Moreover, we try to find some conditions of minC_{11} (maxC_{11}) modules and apply to minC_{11} (maxC_{11}) rings.

Expected Benefits for the Study

For this study, the definition of minC_{11} and maxC_{11} modules will be defined. Any direct sum of modules with minC_{11} (maxC_{11}) satisfies minC_{11} (maxC_{11}) will be proved. The conditions to the direct summand of minC_{11} module satisfies minC_{11} will be obtained. Finally, pure mathematical research helps us to improve and refresh the quality of what we teach, and certainly the world needs a large number of graduates with a wide variety of mathematical skills to fill the wide variety of positions that require some mathematics or the ability to analyze problems logically.

CHAPTER 2

REVIEW OF LITERATURE

In this chapter, we investigate some fundamental properties of *CS* modules and study the direct sum of minC_{11} and maxC_{11} modules. Moreover, their related results are stated. Therefore, for this study some useful definitions and theorems will be presented as follows.

Literature Review

Mohamed and Muller (1990 : 12-37) introduce the extending module defied by a *R*-module *M* is called an extending (or, *CS*), if every submodule is essential in a direct summand of *M*. Equivalently, *M* is extending, if and only if every closed submodule is a direct summand. Let *M* be a right *R*-module. We consider the following conditions.

(C_1) Every submodule of *M* is essential in a direct summand of *M*.

(C_2) Every submodule of *M* which is isomorphic to a direct summand of *M* is itself a direct summand of *M*.

(C_3) For any direct summands M_1, M_2 of M such that $M_1 \cap M_2 = 0$, the submodule $M_1 \oplus M_2$ is also a direct summand of *M*.

M is called *continuous* if it satisfies conditions (C_1) and (C_2) ; *quasi continuous* if it satisfies conditions (C_1) and (C_3) ; CS-module if it satisfies only the conditions (C_1).

From the above conditions, we have :

injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS

Dung, Huynh, Smith, and Wisbauer (1994 : 55-65) studied extending module and found many properties of extending modules. The interesting properties of extending module that is any direct summand of an extending module is also extending. In particular, for any ring *R*, π-injective *R*-modules are extending. Moreover, let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ be a finite direct sum of relatively injective module M_i, then *M* is extending if and only if all M_i are extending.

Smith and Tercan (1993 : 1809-1847) defined C_{11} module as follows. An R-module M is called a C_{11} module, if every submodule of M has a complement which is a direct summand of M, i.e., for each submodule N of M there exists a direct summand K of M such that K is a complement of N in M. C_{11} modules was defined as a general of CS modules. They studied C_{11} modules and found many properties of C_{11} modules as follows.

Any direct sum of modules with C_{11} satisfies C_{11}. Moreover, a module M satisfies C_{11} if and only if $M = Z_2(M) \oplus K$ for some nonsingular submodule K of M and both $Z_2(M)$ and K satisfies C_{11}.

Husain (2005 : 13-53) introduce the concept of minCS and maxCS modules, where an R-module M is called minCS (maxCS) if every minimal submodule (every maximal submodule with nonzero right annihilator) is a direct summand of M. This result, in particular, Hadi and Majeed proved that, if R is a nonsingular ring then R is a maxCS ring if and only if R is a minCS ring. Later, they proved that a direct summand of minCS (maxCS) modules is minCS (maxCS) modules, but the converse is not true, in general. Moreover, Thuat, Hai, Nghiem and Chairat, proved that if M is semiprime, weak duo module, then M is maxCS if and only if it is minCS. In addition, Jain, Al-Hazmi and Alahmadi, proved that if R is a prime ring which is not a domain, then R is a right nonsingular, right max-min CS with uniform right ideal if and only if R is left nonsingular, left max-min CS with uniform left ideal.

Barnard (1981 : 174-178) defined multiplication modules and residual in a ring as follows.

A right R-module M is called a *multiplication* modules if every submodule of M is of the from MI, for some ideal I of R. Let N be a submodule of M of an R-module M, the ideal

$$(N : M_R) = \{r \in R \mid Mr \subseteq N\}$$

is called *residual* of N by M in R and $(0 : M_R)$ is called *annihilator* of M.

Hadi and Majeed (2012a : 1-13) studied multiplication and proved their theorems as follows.

In commutative ring R, if M is a faithful, finitely generated, and multiplication R-module, then M is minCS (maxCS)-modules if and only if R is minCS (maxCS)-rings.

Theoretical Background

Definitions and theorems

For basic definitions, theorems and notations that will be appeared in this study, we refer to Smith and Tercan (1993), Mohamed and Muller (1990), Tercan and Yucel (2016), Kasch (1982), Lam (1991), Husain (2005), and Dung, Huynh, Smith and Wisbauer (1994). However, many of them can also be found in other texts on modules and rings theory, e.g. Anderson-Fuller (1992), Faith (1973) and Passman (1991). Here we recall some notations which are used for investigations presented in this study.

Definition 2.1 A *ring* is a non-empty set R together with two binary operations, that we shall denote by $+$ and $\cdot$ and called *addition and multiplication* (also called *product*), respectively, such that, for all $a, b, c \in R$ the following axioms are satisfies.

(1) $(R, +)$ is an additive Abelian group.

(2) $(R, \cdot)$ is a multiplicative semi group.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R,\ a\cdot(b+c)=(a\cdot b)+(a\cdot c),\ (a+b)\cdot c=(a\cdot c)+(b\cdot c).$

(The two distributive law are respectively called the *left distributive law* and the *right distributive law*.) We shall usually write simply ab instead of $a\cdot b$ for $a, b \in R.$

Definition 2.2 An *associative ring* is a ring R in which multiplication is associative; that is, for all $a, b, c \in R,\ (a\cdot b)\cdot c = a\cdot(b\cdot c).$ Our rings will be associative rings.

Definition 2.3 A *ring with identity* is a ring R in which the multiplicative semi group has an identity element; that is, there exists $e \in R$ such that $ae = a = ea$ for all $a \in R.$ The element e is called the *identity* or *unity* element of R. Generally, the identity element is denote by 1.

Definition 2.4 A *commutative ring* is a ring R in which multiplication is commutative; that is, $ab = ba$ for all $a, b \in R.$

Throughout, all ring are associative rings with identity unless otherwise stated.

Definition 2.5 Let $(R, +, \cdot)$ be a ring and let S be a non-empty subset of R. Then S is called a *subring* of R if $(S, +, \cdot)$ itself is a ring.

Definition 2.6 A non-empty subset I of R is called a *right ideal* of R if

(1) $a, b \in I$ implie $a - b \in I$, and

(2) $ar \in I$ for all $a \in I$ and $r \in R$.

Definition 2.7 Let I be a right ideal of R.

(1) I is called *maximal* if $I \neq R$ and for any right ideal $J \supseteq I$, either $J = I$ or $J = R$.

(2) I is called *minimal* if $I \neq 0$ and for any right ideal $J \subseteq I$, either $J = I$ or $J = 0$.

Definition 2.8 Let M be an Abilian group with binary operation $+$. Let End$M$ denote the collection of *endomorphism* θ of M, i.e., $\theta : M \to M$ satisfies

$$\theta(a+b) = \theta(a) + \theta(b) \qquad (a, b \in M).$$

Define addition and multiplication in End$M$ by

$$(\theta + \phi)(a) = \theta(a) + \phi(a)$$

$$(\theta \cdot \phi)(a) = \theta(\phi(a))$$

for all $\theta, \phi \in \text{End}M$, $a \in M$. With these definition it can checked that $(End(M), +, \cdot)$ is a ring, celled the *endomorphism ring* of M, with zero element the *zero mapping* $0 : M \to M$ given by $0(m) = 0$ $(m \in M)$ and identity element the *identity mapping* $1 : M \to M$ given $1(m) = m$ $(m \in M)$.

Definition 2.9 Let M be an Abelian group and let R be a ring with 1. Then M is said to be a *right R-module* if and only if there exists a map $M \times R \to M$, written multiplicatively as $(m, r) \mapsto mr$, such that

(1) $(m_1 + m_2)r = m_1 r + m_2 r$,

(2) $m(r_1 + r_2) = mr_1 + mr_2$,

(3) $m(r_1 r_2) = (mr_1)r_2$, and

(4) $m1 = m$

for all $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$. Note that if R is a field, then a right R-module is precisely a right R-vector space.

Throughout, all module are unitary right R-module unless otherwise stated.

Definition 2.10 A non-empty subset N of right R-module M is called *submodule* of M if

(1) for all $a, b \in N$, $a - b \in N$ and

(2) $ar \in N$ for all $a \in N$ and $r \in R$.

Definition 2.11 Let M be an R-module and N be a submodule of M.

(1) N is called a *maximal submodule* of M if $N \neq M$ and for any submodule K of M such that $N \subseteq K$, we have $K = M$ or $K = N$.

(2) N is called a *minimal submodule* of M if $N \neq 0$ and for any submodule K of M such that $K \subseteq N$, we have $K = 0$ or $K = N$.

Definition 2.12 Let X be a subset of R-module M. Then the set

$$N = \left\{ \sum_{i=1}^{n} x_i r_i \mid x_i \in X,\ r_i \in R,\ n \in \mathbb{N} \right\}$$

is a submodule of M and it is called the *submodule of M generated by X* and is denoted by $|X)$. A subset X of a module M is called a *generating set of M* if $|X) = M$.

Definition 2.13 A module (or right ideal) is called finitely generated if and only if it has a finite generating set.

Definition 2.14 An R-module M is called *simple module* if $M \neq 0$ and for any submodule N of M, $N = 0$ or $N = M$. We emphasize in addition that the minimal submodules are precisely simple submodules.

Lemma 2.15 An R-module M is simple $\Leftrightarrow$ $M \neq 0 \wedge \forall m \in M\,[m \neq 0 \Rightarrow mR = M]$.

Proof. (Kasch F. 1982:19)

Definition 2.16 An R-module M is called *cyclic* : $\Leftrightarrow \exists m_0 \in M\,[M = m_0R]$.

Definition 2.17 An R-module M is called the *direct sum* of the set $\{B_i \mid i \in I\}$ of submodules B_i of M, in symbols:

$$M = \bigoplus_{i \in I} B_i \begin{cases} 1)\, M = \sum_{i \in I} B_i, \\ 2)\, \forall j \in I \left[B_j \cap \sum_{i \in I, i \neq J} B_i = 0 \right]. \end{cases}$$

$M = \bigoplus_{i \in I} B_i$ is called a *direct decomposition* of M into the sum of submodules $\{B_i \mid i \in I\}$.

In case of finite index set, say $I = \{1, 2, 3, ..., n\}$. M is also written as

$$M = B_1 \oplus B_2 \oplus ... \oplus B_n = \bigoplus_{i=1}^{n} B_i.$$

Definition 2.18 A submodule N of M is called a *direct summand* of M, denote by $N \subseteq_{\oplus} M$, if there exists a submodule K of M with $M = N \oplus K$. Example, in Z_Z the ideal nZ with $n \neq 0$, $n \neq \pm 1$ is not a direct summand.

Definition 2.19 An R-module $M \neq 0$ is called a *directly indecomposable* if it is not a direct sum of two non-zero submodules. i.e., 0 and M are the only direct summands of

M. Examples, every simple module *M* is directly indecomposable for it has only 0 and *M* as submodules, $\mathbf{Z}_{\mathbf{Z}}$ is a directly indecomposable.

Definition 2.20 Let *N* be a submodule of an *R*-module *M*. We define *factor module* (*or quotient module*) $M / N = \{m + N \mid m \in M\}$, with the addition and multiplication by any elements $m, m_1, m_2 \in M$ and $r \in R$ by setting,

(1) $(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N$ and

(2) $(m + N)r = mr + N$.

Note that the factor module has a natural map $\varphi : M \to M / N$ define by $m \mapsto m + N$. This natural map is called *natural* (*canonical*) *epimorphism* of *M* to the factor module M / N. Moreover, it is easy to see that φ is epimorphism.

Definition 2.21 Let *R* be a ring and *M* be an *R*-module. The following are given

(1) a submodule *N* of *M* is called *essential* (or *large*) in *M*, denote by $N \subseteq_e M$, if *N* has non-zero intersection with any non-zero submodule of *M*. If *N* is essential in *M*, we say that *M* is *essential extension* of *N*. Clearly, $M \subseteq_e M$.

(2) a submodule *N* of *M* is called *complement* to the submodule *K* of *M*, if *N* is maximal with respect to property that $N \cap K = 0$. A submodule *N* of *M* will be called complement in *M*, provided there exists $K \subseteq M$ such that *N* is a complement of *K* in *M*. By Zorn's Lemma, any submodule of *M* has a complement.

(3) a submodule *N* of *M* is called *closed* in *M*, denote $N \subseteq_{cl} M$, if it has no proper essential extension in *M*. i.e., if $K \subset M$ such that $N \subseteq_e K$, then $N = K$. Closed submodule are precisely complement submodule (Husain. 2005:14).

Theorem 2.22 Every submodule in M is a direct summand if and only if every submodule is closed.

Proof. (Kasch. 1982:139).

Definition 2.23 An *R*-module *M* is called a *uniform module* if $M \neq 0$ and any two non-zero submodules of *M* intersect nontrivially (equivalently: any non-zero submodule of M indecomposable, or else: any non-zero submodule of M is essential in M). Clearly, uniform closed submodule of M are precisely minimal closed submodule of M.

Definition 2.24 A *right annihilator* of M in R, denote by $ann_R(M)$, is the set of all elements in R such that, for all $m \in M$, $mr = 0$.

Definition 2.25 An R-module M is called a *faithful module* if its $ann_R(M) = 0$.

Definition 2.26 An R-module M is called an extending module (or CS-module) if every submodule is an essrntial in a direct summand of M. Equivalently, M is extending, if and only if every closed submodule is a direct summand.

Definition 2.27 An R-module M is called a C_{11} module, if every submodule of M has a complement which is a direct summand of M. i.e., for each submodule N of M, there exists a direct summand K of M such that K is a complement of N in M.

Definition 2.28 An R-module M is called a minCS module if every minimal submodule is a direct summand of M.

Definition 2.29 An R-module M is called a maxCS module if every maximal submodule with nonzero right annihilator is a direct summand of M.

Proposition 2.30 Let $N \subseteq M$. There exists $K \subseteq M$, containing N, such that $N \subseteq_e K \subseteq_c M$.

Proof. (Tercan and Yucel. 2016:76).

Definition 2.31 Let M is called a *self-generator* if it generates all its submodules.

Lemma 2.32 Let M be a finitely generated, quasi-projective right R-module which is a self-generator and S its endomorphism ring. Then X is a direct summand of M if and only if $I_X = \{f \in S \mid f(M) \subseteq X\}$ is a direct summand of *S*. In this case, $X = (M)e$ and $I_X = Se$ for some idempotent $e \in S$.

Proof. (Thuat, Hai, Nghiem, and Chairat. 2016:3)

Lemma 2.33 Let *M* be a finitely generated, quasi-projective right *R*-module which is a self-generator and *S* its endomorphism ring. Then

(1) X is a maximal submodule of *M* if and only if $I_X = \{f \in S \mid f(M) \subseteq X\}$ is a maximal right ideal of *S*.

(2) Conversely, *K* is a maximal right ideal of *S* if and only if $KM = \sum_{s \in K} s(M)$ is a maximal submodule of *M*.

Proof. (Thuat, Hai, Nghiem, and Chairat. 2016:3)

Lemma 2.34 Let *M* be a finitely generated, quasi-projective right *R*-module which is a self-generator and *S* its endomorphism ring. Then

(1) X is a minimal submodule of *M* if and only if $I_X = \{f \in S \mid f(M) \subseteq X\}$ is a minimal right ideal of *S*.

(2) Conversely, *K* is a minimal right ideal of *S* if and only if $KM = \sum_{s \in K} s(M)$ is a minimal submodule of *M*.

Proof. (1) Let X is a minimal submodule of *M* and let $I_X \{f \in S \mid f(M) \subseteq X\}$ which is a right ideal of *S*. By hypothesis $I_X \neq 0$. Suppose that, there exists a non-zero right ideal *J* of *S* such that $0 \neq J \subseteq I_X$. Then we have $0 \neq JM \subseteq I_X M = X$, since *M* is a self-generator. Hence $JM = I_X M$. This implies that $J = I_X$. Therefore, I_X is a minimal right ideal of *S*.

Conversely, let $I_X \{f \in S \mid f(M) \subseteq X\}$ be a right ideal of *S* and let X is a minimal submodule of *M*. Then $X \neq 0$. Suppose that, there exists a non-zero submodule *N* of *M* such that $0 \neq N \subseteq X$. Then we have $0 \neq N = I_N M \subseteq \ X = I_X M$, since *M* is a self-generator. Hence $I_N = I_X$. This implies that $N = X$. Therefore, X is a minimal right ideal of *S*.

(2) We use the same argument as that given in (1).

On faithful multiplication modules

Throughout, this section all ring will be commutative ring with identity and all right R-module will be unitary.

Definition 2.35 Let R be a ring and N and K be submodules of an R-module M, the set $(N:K)=\{r\in R \mid Kr\subseteq N\}$ is called *residual of* N *by* K *in* R and it is an ideal of R, and for every ideal I of R, the set $(N:I)=\{m\in M \mid mI\subseteq N\}$ is called *residual of* N *by* I *in* M and it is a submodule of M.

Definition 2.36 An R-module M is called *multiplication module* if for each N of M, there exists an ideal I of R such that $N=MI$.

Proposition 2.37 If M is a faithful multiplication R-module, then the following statements are equivalent.

(1) M is finitely generated.

(2) If A and B are ideals of R such that $MA\subseteq MB$ then $A\subseteq B$.

(3) For each submodule N of M there exists a unique ideal I of R such that $N=MI$.

(4) $M\neq MA$ for any proper ideal A of R.

(5) $M\neq MP$ for any maximal ideal P of R.

Proof. (El-Bast and Smith. 1988:768).

Proposition 2.38 Let M be a faithful R-module. Then M is a multiplication module if and only if

(1) $\bigcap_{\lambda\in\Lambda}(MI_\lambda)=M(\bigcap_{\lambda\in\Lambda}I_\lambda)$ for any non-empty collection of ideals I_λ, $\lambda\in\Lambda$ of R, and

(2) for any submodule N of M and ideal A of R such that $N\subsetneq MA$ there exists an ideal B with $B\subsetneq A$ and $N\subseteq MB$.

Proof. (El-Bast and Smith. 1988:759).

Proposition 2.39 If M is a faithful multiplication R-module, then M is a finitely generated.

Proof. (Lee D. and Lee H. 1993:133)

Proposition 2.40 Let M be a faithful multiplication R-module. Then a submodule N of M is maximal if and only if there exists a maximal ideal I of R such that $N = MI$.

Proof. Suppose that N is a maximal submodule of M. Then, there exists an ideal I of R such that $N = MI$. It is sufficient to prove that I is maximal ideal of R. For any ideal J of R such that $I \subseteq J \subseteq R$, $N = MI \subseteq MJ \subseteq MR = M$. Since N is a maximal submodule of M, either $MJ = MI$ or $MJ = M$. If $MJ = MI$ then $J = I$ by Proposition 2.37 (2). If $MJ = M$ then $J = R$, again by Proposition 2.37 (2). Therefore, I is a maximal ideal of R.

Conversely, suppose that $N = MI$ for some maximal ideal I of R. Let X be a submodule of M such that $N \subseteq X \subseteq M$. Thus

$$I = (MI : M) = (N : M) \subseteq (X : M) \subseteq R.$$

Since I is a maximal ideal of R, either $(X : M) = I$ or $(X : M) = R$. If $(X : M) = I$ then $X = M(X : M) = MI = N$. If $(X : M) = R$ then $X = M(X : M) = MR = M$. This show that N is a maximal submodule of M.

Proposition 2.41 Let M be a faithful multiplication R-module. Then a submodule N of M is minimal if and only if there exists a minimal ideal I of R such that $N = MI$.

Proof. Suppose that N is a minimal submodule of M. Then, there exists an ideal I of R such that $N = MI$. It is sufficient to prove that I is minimal ideal of R. For any ideal J of R such that $J \subseteq I$, we let $JM \subseteq MI = N$. By hypothesis, either $MJ = 0$ or $MJ = MI$. If $MJ = 0$ then $J = 0$ because M is faithful. If $MJ = MI$ then $J = I$, by Proposition 2.37 (2). Therefore, I is a minimal ideal of R.

Conversely, suppose that $N = MI$ for some minimal ideal I of R. Let X be a submodule of M such that $X \subseteq N$. Then $(X : M) \subseteq (N : M) = (MI : M) = I$. By assumption, either $(X : M) = 0$ or $(X : M) = I$. If $(X : M) = 0$. Thus $X = M(X : M) =$

0. If $(X:M)=I$, then $X=M(X:M)=MI=N$. This shows that N is a minimal submodule of M.

Proposition 2.42 Let M be a faithful multiplication R-module and I, J be ideals of R. Then, $M=MI\oplus MJ$ if and only if $R=I\oplus J$.

Proof. Assume that $M=MI\oplus MJ$. Since $M=MI+MJ$, we have $MR=M=MI+MJ=(I+J)M$ by Proposition 2.39 and Proposition 2.37 (2). Thus $R=I+J$, and $M(I\cap J)=MI\cap MJ=0$ by Proposition 2.38 (1), which implies that $I\cap J=0$. Therefore, $R=I\oplus J$.

Conversely, suppose that $R=I\oplus J$, where I and J are ideals of R. Then MI and MJ are submodules of M. Thus $M=MR=M(I+J)=MI+MJ$ and by Proposition 2.38 (1), we have $MI\cap MJ=M(I\cap J)=0$. Therefore, $M=MI\oplus MJ$.

CHAPTER 3

MINC_{11} AND MAXC_{11} MODULES

Throughout this chapter, all rings well be associative ring with identity and all right modules will be unitary. In this chapter, we will define of minC_{11} and maxC_{11} modules and find some related basic results.

Definitions and Examples

In this section we well introduce the notion of minC_{11} and maxC_{11} modules with some examples.

Definition 3.1 An R-module M is said to be minC_{11} module, if every minimal submoodule has a complement which is a direct summand of M. i.e., for each minimal submodule N of M there exists a direct summand K of M such that K is a complement of N in M. A ring R is minC_{11} if it is minC_{11} R-module.

Definition 3.1 An R-module M is said to be maxC_{11} module, if every maximal submoodule with nonzero right annihilator has a complement which is a direct summand of M. i.e., for each minimal submodule L of M with nonzero right annihilator there exists a direct summand K of M such that K is a complement of L in M. A ring R is maxC_{11} if it is maxC_{11} R-module.

Remarks and Examples

(1) Every C_{11}-module is minC_{11} and maxC_{11}. Because any submodule has a complement which is a direct summand. But convert is not true in general.

(2) Every CS-module is minC_{11} and maxC_{11}.

Proof. By (Smith and Tercan. 1993:1814), every CS-module is C_{11}.

(3) Every simple module is minC_{11} and maxC_{11}. In particular, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_6$, $\mathbf{Z}_{10}$ as a $\mathbf{Z}$-module is minC_{11} and maxC_{11}.

Proof. By (Dung, Huynh, Smith and Wisbauer. 1994:55), every simple module is CS.

(4) Every uniform module is minC_{11} and maxC_{11}. In particular, each of $\mathbf{Z}$-module $\mathbf{Z}$, $\mathbf{Z}_4$, $\mathbf{Z}_8$, $\mathbf{Z}_9$, $\mathbf{Z}_{16}$ is minC_{11} and maxC_{11}.

MinC_{11} and MaxC_{11} Modules Properties

In this section, we give preliminary results which will be used in the later chapters. We stat this section by a simple and useful result.

Lemma 3.3 Let N be a submodule of M and K be a direct summand of M. K is a complement of N in M if and only if $K \cap N = 0$ and $K \oplus N \subseteq_e M$.

Proof. Suppose K is a complement of N in M. Then $K \cap N = 0$. Let $0 \neq x \in M$. If $x \in K$, then $0 \neq xR = xR \cap K \subseteq xR \cap (K \oplus N)$. If $x \notin K$, then $N \cap (xR + K) \neq 0$ and hence $xR \cap (K \oplus N) \neq 0$. Then $xR \cap (K \oplus N) \neq 0$ for all $0 \neq x \in M$. Thus $K \oplus N \subseteq_e M$.

Conversely, suppose that K and N have the stated properties. There exists a submodule K' of M such that $M = K \oplus K'$. Suppose that there exists a submodule K_1 of M such that $K \subseteq K_1$ and $K_1 \cap N = 0$. Then $K_1 = K_1 \cap M = K_1 \cap (K \oplus K') = K \oplus (K_1 \cap K')$. Let $0 \neq y \in (K_1 \cap K')$. therefore, $0 \neq yr = n + k$ for some $n \in N$, $k \in K$, $r \in R$. $yr - k = n \in K_1 \cap N = 0$. Thus $yr = k \in K \cap K' = 0$, a contradiction. Hence $K_1 \cap K' = 0$ and $K_1 = K$. That is, K is a complement of N in M.

Proposition 3.4 An R-module M satisfies minC_{11} if and only if for any minimal submodule N of M, there exists a direct summand K of M such that $K \cap N = 0$ and $K \oplus N \subseteq_e M$.

Proof. Let N be a minimal submodule of M. By hypothesis, there exists a direct summand K of M such that K is a complement of N in M. By Lemma 3.2, $K \cap N = 0$ and $K \oplus N \subseteq_e M$.

Conversely, suppose that K and N have the stated properties. By Lemma 3.2, K is a complement of N in M. Hence M satisfies minC_{11}.

Proposition 3.5 An R-module M satisfies maxC_{11} if and only if for any maximal submodule L of M with non-zero right annihilator, there exists a direct summand K of M such that $K \cap L = 0$ and $K \oplus L \subseteq_e M$.

Proof. Let N be a maximal submodule of M with non-zero right annihilator. By hypothesis, there exists a direct summand K of M such that K is a complement of L in M. By Lemma 3.2, $K \cap L = 0$ and $K \oplus L \subseteq_e M$.

Conversely, suppose that M satisfies the stated conditions. By Lemma 3.2, K is a complement of L in M. Hence M satisfies minC_{11}.

Lemma 3.6 Let M be an R-module and I be an ideal of R such that $I \subseteq Ann_R(M)$. Then

(1) A submodule N be a minimal submodule of R-module if and only if N be a minimal of (R/I)-module.

(2) A submodule L be a maximal submodule of R-module if and only if L be a maximal of (R/I)-module.

Proof. (1) Suppose N be a minimal submodule of R-module. Let $0 \neq m \in N$, then $m(r+I) = mr + mI = mr, \forall (r+I) \in (R/I), m \in M$. By Lemma 2.13, N is a minimal (R/I)-module. Conversely, Suppose N is a minimal submodule of (R/I)-module. Let $0 \neq m \in N$, then $mr = mr + 0 = mr + mI = m(r+I), \forall r \in R, m \in M$. By Lemma 2.13, N is a minimal R-module.

(2) Suppose L be a maximal submodule of R-module M. Then M/L is simple submodule of R-module. By (1), M/L is simple submodule of (R/I)-module. Hence L is a maximal submodule of (R/I)-module. Conversely, let L is a maximal submodule of (R/I)-module M. Then M/L is simple submodule of (R/I)-module. By (1), M/L is simple submodule of R-module. Hence L is a maximal submodule of R-module.

Proposition 3.7 Let M be an R-module and let I be an ideal of R such that $I \subseteq Ann_R(M)$. M is maxC_{11} R-module then M is maxC_{11} (R/I)-module and the converse is true if $Ann_R(M) \neq Ann_R(L)$.

Proof. Let L be a maximal submodule of (R/I)-module and $Ann_{R/I}(L) \neq 0_{R/I} = I$. By Lemma 2.6, L is a maximal submodule of R-module. Since $Ann_{R/I}(M) \neq I = 0_{R/I}$, so there exists $r + I \in R/I$ with $r \notin I$ such that $r + I \in Ann_{R/I}(L)$, hence $r \neq 0$ and $Nr = 0$. Thus $Ann_R(L) \neq 0$. By hypothesis, there exists a direct summand K of M_R such that K is a complement of L in M_R. It is easy to see that K is a complement of L in $M_{R/I}$ and K is a direct summand in $M_{R/I}$. That is, M is maxC_{11} (R/I)-module.

Conversely, let L be a maximal R-module with $Ann_R(L) \neq 0$. Then L is a maximal (R/I)-module. Now, since $Ann_R(M) \neq Ann_R(L)$, there exists $r \in Ann_R(L)$ and $r \notin Ann_R(M)$. Thus $r \notin I$, that is $0_{R/I} = I \neq r + I$ and $L(r + I) = 0$. Hence $Ann_{R/I}(L) \neq 0$. But M is a maxC_{11} (R/I)-module, there exists a direct summand K of $M_{R/I}$ such that K is a complement of L in $M_{R/I}$. Therefore, that K is a complement of L in M_R and K is a direct summand in M_R. That is, M is maxC_{11} R-module.

Proposition 3.8 Let M be an R-module and let I be an ideal of R such that $I \subseteq Ann_R(M)$. M is minC_{11} R-module if and only if M is minC_{11} (R/I)-module.

Proof. Similar Proposition 2.7.

Proposition 3.9 Let be an R-module.

(1) If M satisfies maxCS then M satisfies maxC_{11}.

(2) If M satisfies minCS then M satisfies minC_{11}.

Proof. (1) Clear.

(2) Let M is a minCS R-module and let N be a minimal submodule of M. By Proposition 2.29, there exists a complement submodule K of M, contain N, such that $N \subseteq_e K \subseteq_c M$. It is easy to see that K is a minimal closed submodule of M. By hypothesis, K is a direct summand of M. Let $M = K \oplus K'$ for some submodule K' of M. It is clearly that $N \cap K' = 0$ and $N \oplus K' \subseteq_e M$. By Lemma 3.3, K' is a complement of N in M. That is, M satisfies minC_{11}.

CHAPTER 4

ON THE DIRECT SUM OF MINC_{11} AND MAXC_{11} MODULES

Throughout this chapter, all rings are associative with identity and all right modules are unitary. In this chapter, we studied direct sums of minC_{11} and maxC_{11} modules and find out further properties.

The direct sum of MinC_{11} and MaxC_{11} Modules

Lemma 4.1 Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$, such that each M_λ satisfies minC_{11}, if N be a minimal submodule of M then there exists a unique M_λ with $N \cap M_\lambda \neq 0$.

Proof. Let N be a minimal submodule of M. Then there exists a M_λ of M such that $N \cap M_\lambda \neq 0$. Next we show that has unique. Suppose there exists a $M_\gamma \neq M_\lambda$, $\exists \gamma \in \Lambda$, such that $N \cap M_\gamma \neq 0$. Hence by property of N, we have $0 \neq N = N \cap M_\gamma \subseteq M_\gamma$, a contradiction. Therefore, $M_\lambda = M_\gamma$.

Theorem 4.2 Any direct sum of module with minC_{11} satisfies minC_{11}.

Proof. Let M_λ $(\lambda \in \Lambda)$ be non-empty collection of module, each satisfies minC_{11}. Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Let N be a minimal submodule of M. By Lemma 4.1, there exists a unique M_λ of M such that $0 \neq N = N \cap M_\lambda$. Since M_λ satisfies minC_{11}, then there exists a direct summand K_λ of M_λ such that $K_\lambda \cap N = 0$ and $K_\lambda \oplus N \subseteq_e M_\lambda$. Let $M' = \oplus_{\gamma\in\Lambda,\, \gamma\neq\lambda} M_\gamma$. It is clearly that $M' \cap N = 0$ and $K_\lambda \cap M' = 0$. Let $K = K_\lambda \oplus M'$, then $N \cap K = 0$. Next we show that $N \oplus K \subseteq_e M$. Let $0 \neq A \subseteq M$. If $A \cap N \neq 0$, then $0 \neq A \cap N \subseteq A \cap (N \oplus K)$. If $A \cap N = 0$, then we shall shown that $A \cap K \neq 0$. Suppose that $A \cap K = 0$. then $A \cap K = A \cap (K_\lambda \oplus M') = (A \cap K_\lambda) \oplus (A \cap M') = 0$. Thus $A = 0$, a contradiction. Then $0 \neq A \cap (N \oplus K)$. Hence $N \oplus K \subseteq_e M$. That is, M satisfies minC_{11}

Lemma 4.3 Let $M=\oplus_{\lambda\in\Lambda}M_\lambda$, such that each M_λ satisfies maxC_{11}. If L is any maximal submodule of M, then there exists at least one M_λ with $L\cap M_\lambda\neq M_\lambda$.

Proof. Let L be any maximal submodule of M. If $L\cap M_\lambda=M_\lambda$, for all $M_\lambda\subseteq M$. Then $L=\oplus_{\lambda\in\Lambda}M_\lambda=M$, a contradiction. Therefore, there exists at least one M_λ with $L\cap M_\lambda\neq M_\lambda$.

Theorem 4.4 Any direct sum of modules with maxC_{11} satisfies maxC_{11}.

Proof. Let M_λ $(\lambda\in\Lambda)$ be a non-empty collection of module, each of them satisfies maxC_{11}. Let $M=\oplus_{\lambda\in\Lambda}M_\lambda$. Let L be a maximal submodule of M. By property of L, there exists at least one M_λ such that $L\cap M_\lambda\neq 0$. Hence $L\cap M_\lambda$ is a maximal submodule of M_λ. But M_λ satisfies maxC_{11}, by Proposition 3.5, there exists a direct summand K_λ of M_λ such that $K_\lambda\cap(L\cap M_\lambda)=0$ and $K_\lambda\oplus(L\cap M_\lambda)\subseteq_e M_\lambda$. Note that $L\cap K_\lambda=0$, $(K_\lambda\oplus L)\cap M_\lambda=K_\lambda\oplus(L\cap M_\lambda)$ and $(K_\lambda\oplus L)\cap M_\lambda\subseteq_e M_\lambda$. Let Λ' be non-empty subset of Λ containing λ such that there exists a direct summand K' of $M'=\oplus_{\lambda\in\Lambda'}M_\lambda$ with $L\cap K'=0$ and with $(L\oplus K')\cap M'\subseteq_e M'$. Suppose $\Lambda'\neq\Lambda$. Let $\mu\in\Lambda$, $\mu\notin\Lambda'$. Let $\Lambda''=\Lambda'\cup\{\mu\}$ and $M''=\oplus_{\lambda\in\Lambda''}M_\lambda=M'\oplus M_\mu$. Now $A=(L\oplus K')\cap M_\mu$ is a submodule of M_μ.

If $A=M_\mu$, we have $M_\mu\subseteq L$. Then $L\cap M''=(L\cap M_\lambda)\oplus M_\mu$ is a maximal submodule of M'', and $Ann_R(L\cap M'')\neq 0$. Let $K''=K'$ is a submodule of M'' then K'' is a direct summand of M'' and moreover $(L\cap M'')\cap K''=0$. Consider the submodule $L\oplus K''$. Note that $(L\oplus K'')\cap M'=(L\oplus K')\cap M'$ which is an essential submodule of M'. Then $(L\oplus K'')\cap M'\subseteq_e M'$. Moreover $(L\oplus K'')\cap M_\mu=A=M_\mu$ is an essential submodule of M_μ. Hence, $(L\oplus K'')\cap M''\subseteq_e M''$. Therefore, by Lemma 3.3, K'' is a complement of $L\cap M''$ in M''.

If $A\neq M_\mu$, we have A is a maximal submodule of M_μ and $Ann_R(A)\neq 0$. By hypothesis, there exists a direct summand K_μ of M_μ such that $A\cap K_\mu=0$ and

$A \oplus K_\mu \subseteq_e M_\mu$. Since $K_\mu \cap K' = 0$. Let $K'' = K' \oplus K_\mu$. Then K'' is a direct summand of M''. Clearly, $L \cap M'' = (L \cap M') \oplus A$ is a maximal submodule of M'' such that $Ann_R(L \cap M'') \neq 0$, and $(L \cap M'') \cap K'' = 0$. Next we shall show that $(L \oplus K'') \cap M'' \subseteq_e M''$. Consider the submodule $L \oplus K''$. Note that $(L \oplus K'') \cap M'$ contains $(L \oplus K') \cap M'$, so that $(L \oplus K'') \cap M' \subseteq_e M'$. Moreover $(L \oplus K'') \cap M_\mu =$ $(L \oplus K' \oplus K_\mu) \cap M_\mu = [(L \oplus K') \cap M_\mu] \oplus K_\mu = A \oplus K_\mu$, which is an essential submodule of M_μ. Therefore $(L \oplus K'') \cap M'' \subseteq_e M''$. By Lemma 3.3, K'' is a complement of $L \cap M''$ in M''.

Repeating this argument, there exists a direct summand K of M such that $L \cap K = 0$ and $L \oplus K \subseteq_e M$. By Proposition 3.5, M satisfies maxC_{11}.

Corollary 4.5 Any direct summand of modules with minC_{11} (maxC_{11}) satisfies minC_{11} (maxC_{11}).

Proof. Immediate by Theorem 4.2, 4.4.

Corollary 4.6 Any direct sum of C_{11}-modules satisfies minC_{11} and maxC_{11}.

Proof. Immediate by Theorem 4.2 and 4.4.

Corollary 4.7 Any direct sum of *CS*-modules satisfies minC_{11} and maxC_{11}.

Proof. Immediate by Theorem 4.2 and 4.4.

Corollary 4.8 Any direct sum of min*CS*-modules satisfies minC_{11}.

Proof. Immediate by Proposition 3.9 (2) and Theorem 4.2.

Corollary 4.9 Any direct sum of max*CS*-modules satisfies maxC_{11}.

Proof. Immediate by Proposition 3.9 (1) and Theorem 4.4.

Example 4.10 In [2] is show that $M = \mathbf{Z}_2 \oplus \mathbf{Z}_8$ is not min*CS*, but each of $\mathbf{Z}_2$ and $\mathbf{Z}_8$ are minC_{11}. Hence by Theorem 2.10 $M = \mathbf{Z}_2 \oplus \mathbf{Z}_8$ is minC_{11}.

The next results deal with special cases when a direct summand of minC_{11} modules is minC_{11}.

Lemma 4.10 Let $M = M_1 \oplus M_2$. Then M_1 satisfies minC_{11} if and only if for every minimal submodule N of M_1, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$, and $K \oplus N$ is essential submodule of M.

Proof. Suppose M_1 satisfies minC_{11}. Let N be a minimal submodule of M_1. By Proposition 3.4, there exists a direct summand L of M_1 such that $N \cap L = 0$ and $N \oplus L \subseteq_e M_1$. Clearly, $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2) \oplus N$ is an essential in M.

Conversely, suppose M_1 has the stated property. Let H be a minimal submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$, and $K \oplus H$ is an essential submodule of M. Now $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$ so that $K \cap M_1$ is a direct summand of M, and hence also of M_1, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = (H \oplus K) \cap M_1$, which is an essential submodule of M_1. By Proposition 3.4, M_1 satisfies minC_{11}.

Lemma 4.11 Let $M = M_1 \oplus M_2$. Then M_1 satisfies maxC_{11} if and only if for every maximal submodule L of M_1 with nonzero right annihilator, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap L = 0$, and $K \oplus L$ is an essential submodule of M.

Proof. Suppose M_1 satisfies maxC_{11}. Let L be a maximal submodule of M_1 with nonzero right annihilator. By Proposition 3.5, there exists a direct summand N of M_1 such that $L \cap N = 0$ and $L \oplus N \subseteq_e M$. Clearly, $L \cap (N \oplus M_2) = 0$ and $L \oplus (K \oplus M_2)$ is essential in M.

Conversely, suppose M_1 has the stated property. Let H be a maximal submodule of M_1 with nonzero right annihilator. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$, and $K \oplus H$ is an essential submodule of M. Now $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$, hence $K \cap M_1$ is a direct summand of M, and also of M_1, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = (H \oplus K) \cap M_1$, which is an essential submodule of M_1. By Proposition 3.5, M_1 satisfies maxC_{11}.

Theorem 4.12 Let $M = M_1 \oplus M_2$ be a minC_{11}-module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is a minC_{11}-module.

Proof. Let N be a minimal submodule of M_1. By hypothesis, there exists a direct summand K of M such that $(N \oplus M_2) \cap K = 0$, and $N \oplus M_2 \oplus K$ is an essential submodule of M by Proposition 3.4. Moreover, $M_2 \oplus K$ is a direct summand of M. Now the result follows from Lemma 4.11.

Corollary 4.13 Let M be a minC_{11}-module and K is a direct summand of M such that M / K is K-injective. Then K satisfies minC_{11}.

Proof. Let K is a direct summand of M. There exists a submodule K' of M such that $M = K \oplus K'$ and, by hypothesis, K' is K-injective. Let L be a direct summand of M such that $L \cap K' = 0$. By [Dung, Lemma 7.5], there exists a submodule H of M such that $H \cap K' = 0$, $M = H \oplus K'$, and $L \subseteq H$. Thus L is a direct summand of H, hence $L \oplus K'$ is a direct summand of $M = H \oplus K'$. By Theorem 4.12, K satisfies minC_{11}.

Corollary 4.14 Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. Then M satisfies minC_{11} if and only if M_1 satisfies minC_{11}.

Proof. If M satisfies minC_{11}, then M_1 satisfies minC_{11} by Corollary 4.13. Conversely, if M_1 satisfies minC_{11}, then M satisfies minC_{11} by Theorem 4.2.

On the endomorphism rings of MinC_{11} and MaxC_{11} Modules

We close this chapter by considering the relation between minC_{11} (maxC_{11})-modules and their endomorphism rings. Throughout this section, *M* is a right *R*-module with the endomorphism ring *S*. We call *M* a maxC_{11} module if every maximal submodule with nonzero left annihilator has a complement which is a direct summand of *M*. *M* is called a minC_{11} if every minimal submodule has a complement which is a direct summand of *M*. *R* is called a right maxC_{11} (resp. right minC_{11}) ring if R_R is a maxC_{11} (resp. minC_{11}) module.

Theorem 4.15 Let *M* be a finitely generated, quasi-projective right *R*-module which is a self-generator. Then *M* is a maxC_{11} module if and only if *S* is a right maxC_{11} ring.

Proof. We assume that *M* is a maxC_{11}. For every maximal right ideal *K* of *S* with nonzero left annihilator in *S*, KM is a maximal submodule of *M* by Lemma 2.32 (2). Since *K* has nonzero left annihilator, there is some $0 \neq f \in S$ such that $fK = 0$, whence KM has nonzero left nonzero annihilator in *S* (in deed, $fKM = 0$). By hypothesis, there exists a direct summand *X* of *M* such that $KM \cap X = 0$ and $KM \oplus X \subseteq_e M$. Since *X* is a direct summand of *M*, by Lemma 2.31, we have $X = I_X M = eM$ for some idempotent $e \in S$. Consequently, $I_X = eS$ is a direct summand of *S*, and hence $K \cap I_X = 0$. Next we shall show that $K \oplus I_X \subseteq_e S$. Let A be a nonzero right ideal of *S*. Then AM is a submodule of M. Since $(KM \oplus I_X M) \cap AM \neq 0$, this implies $(K \oplus I_X) \cap A \neq 0$, showing that $K \oplus I_X \subseteq_e S$. By Proposition 3.5, *S* is a maxC_{11} ring.

Conversely, let *S* is a right maxC_{11} ring. For an arbitrary maximal submodule *X* of *M* with nonzero left annihilator in *S*, $I_X = \{ f \in S \mid f(M) \subseteq X \}$ is a maximal right ideal of *S* with nonzero left annihilator in *S*. Therefore, there exists a direct summand *K* of *S* such that $I_X \cap K = 0$ and $I_X \oplus K \subseteq_e S$. Since K is a direct summand of *S*, by Lemma 2.31, we have $K = eS$ for some idempotent $e \in S$. Consequently, $KM = eM$ is a direct summand of *M*, and hence $X \cap KM = 0$. Next we shall show that $X \oplus KM \subseteq_e M$. Let *Y* be a nonzero submodule of *M*, then I_Y is a right ideal of S.

Since $(I_X \oplus K) \cap I_Y \neq 0$, this implies $(I_X M \oplus KM) \cap I_Y M \neq 0$, showing that $I_X \oplus K \subseteq_e S$. By Proposition 3.5, S is a maxC_{11} ring.

Theorem 4.15 Let M be a finitely generated, quasi-projective right R-module which is a self-generator. Then M is a minC_{11} module if and only if S is a right minC_{11} ring.

Proof. Similar to that of Theorem 4.14.

Theorem 4.16 Let R be commutative ring. If M is a faithful, finitely generated, and multiplication R-module, then M is a maxC_{11}-module if and only if R is a maxC_{11} ring.

Proof. Let M be a maxC_{11}-module and I be a maximal ideal of R with nonzero annihilator. Hence by Proposition 2.39, MI is a maximal submodule of M. But M is faithful multiplication, we have $Ann(MI) = Ann(I) \neq 0$. Thus by hypothesis, there exists a direct summand K of M such that $MI \cap K = 0$ and $MI \oplus K \subseteq_e M$. Since M is multiplication module, we have $K = MJ$ for some ideal J of R, so by Proposition 2.42, J is a direct summand in R. It is easy to see that $I \cap J = 0$. Next we shall show that $I \cap J \subseteq_e R$. Let A be a nonzero ideal of R, then MA is submodule of M. Since $(MI \oplus MJ) \cap MA = M((I \oplus J) \cap A) \neq 0$, thus $(I \oplus J) \cap A \neq 0$. So that $I \oplus J \subseteq_e R$. By Proposition 3.5, R is maxC_{11} ring.

Let R is a maxC_{11}-ring and L be a maximal submodule of M with nonzero annihilator. Hence by Proposition 2.39, there exists a maximal ideal I of R such that $L = MI$. But M is faithful multiplication, we have $Ann(L) = Ann(I) \neq 0$. Thus by hypothesis, there exists a direct summand J of R such that $I \cap J = 0$ and $I \oplus J \subseteq_e R$. Since M is a multiplication module, we have $K = MJ$ is a submodule of M. So by Proposition 2.42, K is a direct summand in M. It is easy to see that $L \cap K = 0$. Next we show that $L \cap K \subseteq_e M$. Let N be a nonzero submodule of M. We have $N = MA$ for some ideal A of R. Since $(L \oplus K) \cap N = M((I \oplus J) \cap A) \neq 0$, we get $(L \oplus K) \cap N \neq 0$. Thus $L \oplus K \subseteq_e M$. By Proposition 3.5, M is a maxC_{11}-module.

Theorem 4.17 In commutative ring R, if M is a faithful, finitely generating, and multiplication R-module, then M is minC_{11}-module if and only if R is minC_{11} ring.

Proof. Similar to that of Theorem 4.16.

CHAPTER 5

CONCLUSIONS

In this study, we proposed and proved the following properties.

1. Definition of MinC_{11} and MaxC_{11} modules

1.1 An R-module M is said to be a minC_{11} module, if every minimal submodule has a complement which is a direct summand of M. i.e., for each minimal submodule N of M there exists a direct summand K of M such that K is a complement of N in M. A ring R is minC_{11} if it is a minC_{11} R-module.

1.2 An R-module M is said to be a maxC_{11} module, if every maximal submodule with nonzero right annihilator has a complement which is a direct summand of M. i.e., for each minimal submodule L of M with nonzero right annihilator there exists a direct summand K of M such that K is a complement of L in M. A ring R is maxC_{11} if it is a maxC_{11} R-module.

2. MinC_{11} and MaxC_{11} modules properties

2.1 Let N be a submodule of M and K be a direct summand of M. K is a complement of N in M if and only if $K \cap N = 0$ and $K \oplus N \subseteq_e M$.

2.2 An R-module M satisfies minC_{11} if and only if for any minimal submodule N of M, there exists a direct summand K of M such that $K \cap N = 0$ and $K \oplus N \subseteq_e M$.

2.3 An R-module M satisfies maxC_{11} if and only if for any maximal submodule L of M with non-zero right annihilator, there exists a direct summand K of M such that $K \cap L = 0$ and $K \oplus L \subseteq_e M$.

2.4 Let M be an R-module and I be an ideal of R such that $I \subseteq Ann_R(M)$. Then

(1) A submodule N is a minimal submodule of R-module if and only if N is a minimal of (R/I)-module.

(2) A submodule L is a maximal submodule of R-module if and only if L is a maximal of (R/I)-module.

2.5 Let M be an R-module and let I be an ideal of R such that $I \subseteq Ann_R(M)$. M is maxC_{11} R-module then M is a maxC_{11} (R/I)-module and the converse is true if $Ann_R(M) \neq Ann_R(L)$.

2.6 Let M be an R-module and let I be an ideal of R such that $I \subseteq Ann_R(M)$. M is a minC_{11} R-module if and only if M is a minC_{11} (R/I)-module.

2.7 Let be an R-module.

(1) If M satisfies maxCS, then M satisfies maxC_{11}.

(2) If M satisfies minCS, then M satisfies minC_{11}.

3. The direct sum of MinC_{11} and MaxC_{11} modules

3.1 Let $M = \oplus_{\lambda \in \Lambda} M_\lambda$, such that each M_λ satisfies minC_{11}. If N is a minimal submodule of M, then there exists a unique M_λ with $N \cap M_\lambda \neq 0$.

3.2 Any direct sum of modules with minC_{11} satisfies minC_{11}.

3.3 Any direct sum of modules with maxC_{11} satisfies maxC_{11}.

3.4 Any direct summand of modules with minC_{11} (maxC_{11}) satisfies minC_{11} (maxC_{11}).

3.5 Any direct sum of C_{11}-modules satisfies minC_{11} and maxC_{11}.

3.6 Any direct sum of CS-modules satisfies minC_{11} and maxC_{11}.

3.7 Any direct sum of minCS-modules satisfies minC_{11}.

3.8 Any direct sum of maxCS-modules satisfies maxC_{11}.

3.9 Let $M = M_1 \oplus M_2$. Then M_1 satisfies minC_{11} if and only if for every minimal submodule N of M_1, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$, and $K \oplus N$ is an essential submodule of M.

3.10 Let $M = M_1 \oplus M_2$. Then M_1 satisfies maxC_{11} if and only if for every maximal submodule L of M_1 with nonzero right annihilator, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap L = 0$, and $K \oplus L$ is an essential submodule of M.

3.11 Let $M = M_1 \oplus M_2$ be a minC_{11}-module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is minC_{11}-module.

3.12 Let M be a minC_{11}-module and K is a direct summand of M such that M/K is K-injective. Then K satisfies minC_{11}.

3.13 Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. Then M satisfies minC_{11} if and only if M_1 satisfies minC_{11}.

4. The relation between MinC_{11} (MaxC_{11}) Modules and MinC_{11} (MaxC_{11}) Rings

4.1 Let M be a finitely generated, quasi-projective right R-module which is a self-generator. Then M is a maxC_{11} module if and only if S is a right maxC_{11} ring.

4.2 Let M be a finitely generated, quasi-projective right R-module which is a self-generator. Then M is a minC_{11} module if and only if S is a right minC_{11} ring.

4.3 For a commutative ring R, if M is a faithful, finitely generated, and multiplication R-module, then M is a maxC_{11}-module if and only if R is a maxC_{11} ring.

4.4 For a commutative ring R, if M be a faithful, finitely generated, and multiplication R-module, then M is a minC_{11}-module if and only if R is a minC_{11} ring

BIBLIOGRAPHY

Anderson, F. W. and Fuller, K. R. (1992). Rings and Categories of Modules. New York : Springer-Verlag.

Barnard, A. (1981). "Multiplication Modules," Journal of Algebra. 71(1), 174-178.

Bhattacharya, P. B. Jain, S. K. and Nagpaul, S. R. (1994). Basic Abstact Algebra. New York : Cambridge University Press.

Dung, N. V. Huynh, D. V. Smith, P. F. and Wisbauer, R. (1994). Extending Modules. London : Pitman.

El-Bast, Z. A. and Smith, P. F. (1988). "Multiplication Modules," Communication in Algebra. 16(1), 755-779.

Facchini, A. (1998). Module Theory : Endomorphism Ring and direct sum Decomposition in Some Classes of Modules. Basel : Birkhauser.

Faith, C. (1973). Algebra I Rings, Modules and Categories. New York : Springer-Verlag.

Goodearl, K. R. and Warfield, R. B. (1989). An Introduction to Noncommutative Noetherian Rings. New York : Cambridge University Press.

Hadi, I. A. and Majeed, R. N. (2012a). "Max (Min)-CS Modules", Ibn Al-Haitham Journal for Pure and Applied Scinec. 1(25), 1-13.

Hadi, I. A. and Majeed, R. N. (2012b). "On the direct sum of Min (Max)-CS Modules", Juornal of Kufafor Mathematics and Computer. 8(1), 29-36.

Husain, S. A. (2005). A Study of CS and $\Sigma -$CS Ring and Modules. Dissertation Doctor of Science. Ohio : Ohio University.

Kasch, F. (1982). Modules and Rings. London : Academic Press.

Lam, T. Y. (1989). Lectures on Modules and Rings. New York : Springer-Verlag.

Lee, D. and Lee, H. (1993). "Some Remarks on Faithful Multiplication Modules," Journal of the Choungcheong Mathematical Society. 6(1), 131-137.

Mohamed, S. H. and Muller, B. J. (1990). Continuous and Discrete Modules. New York : Cambridge University Press.

Passman, D. S. (1991). A Course in Ring Theory. California : Wadsworth and Book/Cole.

Saeho, K. (2011). On Faithful Multiplication Modules. Master's thesis Master of Science. Songkhla : Thaksin University.

Smith, P. F. and Tercan, A. (1993). "Generalizations of CS Modules," Communication in Algebra. 6(21), 1809-1847.

Tercan, A. and Yucel, C. C. (2016). Module Theory, Extending Modules and Generalization. Basel : Birkhauser Basel.

Thuat, D. V. Hai, H. D. Nghiem, N. D. and Chairat, S. (2016). "On the endomorphism rings of max CS and min CS modules," In AIP Conference Proceedings. pp. 1-7. August 10-12 2016 BP Smila Beach hotel and resort, Songkhla.

Wisbauer, R. (1991). Foundation of Module and Ring Theory : A Handbook for Study and Research. Philadelphia : Gordon and Breach.

APPENDIX

APPENDIX

ACCEPTED PAPER FOR PUBLICATION

IN

PROCEEDING' INTERNATIONAL CONFERENCE ON MATHEMATICS, ENGINEERING & INDUSTRIAL APPLICATIONS 2018

ICoMEIA 2018

Certificate of Participation

This certificate is awarded to

HAGIM RAYALONG

as

PRESENTER

in

International Conference on Mathematics, Engineering & Industrial Applications (ICoMEIA 2018)

on

24-25 July 2018

Organised by

Institute of Engineering Mathematics, Universiti Malaysia Perlis

Co-Organised by

UTM Centre for Industrial & Applied Mathematics, Universiti Teknologi Malaysia, Johor Bahru
and Jabatan Sains Komputer dan Matematik,
Universiti Teknologi MARA, Cawangan Pulau Pinang

PROF. DR. AMRAN AHMED
Chairperson
International Conference on Mathematics,
Engineering & Industrial Applications 2018

ICoMEIA 2018

CURRICULUM VITEA

Student's Name	Mr. Hagim Rayolong
Date of Birth	21 May 1990
Place of Birth	Songkhla, Thailand
Home Address	83, Moo 9, Tambon Paching, Chana District, Songkhla, Thailand, 90130
Position	Teacher
Office	Ban KhlongNgae School, Moo 5, Tambon Phangla Sadao District, Songkhla, Thailand, 90170

Education Background

2014	Bachelor of Education in Mathematics Songkhla Rajabhat University, Songkhla, Thailand
2019	Master of Science in Mathematics and Mathematics Education Thaksin University, Songkhla, Thailand

Research Paper

Rayalong, H. and Chairat, S. (2018) "On the Direct sum of $MinC_{11}$ and $MaxC_{11}$ Modules," in International Conference on Mathematics, Engineering & Industrial Applications 2018. July 24 – 25, 2018 at Menara Razak UTM KL, Malaysia.